Inequality

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Show that for positive real numbers a, b, c, x, y, z

$\sum \frac{a}{b+c}(y+z) \geq 3\left(\frac{xy+yz+zx}{x+y+z}\right)$

and determine when equality holds.

Solution by Arkady Alt , **San Jose**, **California**, **USA**.

For convenience we will use letters α, β, γ instead letters $a, b, c,$ respectively, and now letters a, b, c be free for using their to another work, namely let $a := \sqrt{y+z}$, $b := \sqrt{z+x}, c := \sqrt{x+y}$. Then numbers a, b, c are sidelengths of some acute triangle with area F and since $x = \frac{b^2+c^2-a^2}{2}, y = \frac{c^2+a^2-b^2}{2}, z = \frac{a^2+b^2-c^2}{2}$ then $x+y+z = \frac{a^2+b^2+c^2}{2}$, $xy+yz+zx = \frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4} = 4F^2$ and inequality of the problem becomes $\sum \frac{\alpha}{\beta+\gamma}a^2 \geq \frac{24F^2}{a^2+b^2+c^2}$.

Or, by replacing α, β, γ with x, y, z (which now became free for using) we obtain inequality

(**1**) $\sum \frac{xa^2}{y+z} \geq \frac{24F^2}{a^2+b^2+c^2}$, where $x, y, z > 0$

which is equivalent geometric interpretation of original inequality.

By Cauchy Inequality we obtain $\sum \frac{xa^2}{y+z} = \sum\left(\frac{xa^2}{y+z} + a^2\right) - \sum a^2 =$

$(x+y+z)\sum \frac{a^2}{y+z} - \sum a^2 \geq \frac{(a+b+c)^2}{2} - \sum a^2 = \frac{\Delta(a,b,c)}{2}$,

where $\Delta(a,b,c) := 2ab+2bc+2ca-a^2-b^2-c^2$.

Thus, remains to prove

$\frac{\Delta(a,b,c)}{2} \geq \frac{24F^2}{a^2+b^2+c^2} \Leftrightarrow (a^2+b^2+c^2)\Delta(a,b,c) \geq 48F^2$.

But letter inequality holds because $a^2+b^2+c^2 \geq 4\sqrt{3}F$ (Weitzenböck's inequality) and $\Delta(a,b,c) \geq 4\sqrt{3}F$ is Δ –form of Hadwiger-Finsler Inequality) [1]

Equality in inequality (**1**) holds iff $\frac{y+z}{a} = \frac{z+x}{b} = \frac{x+y}{c}$ (equality condition in Cauchy Inequality) and $a = b = c$ (equality condition in Weitzenböck's and HF inequalities), that is iff $x = y = z$ and $a = b = c$.

Coming back to original inequality we obtain the same conditions of equality in original notations.

1. Arkady Alt, Geometric Inequalities with polynomial $2xy+2yz+2zx-x^2-y^2-z^2$, OCTOGON MATHEMATICAL MAGAZINE vol.22,n.2- 2014.

Link to the article in the comment.